Problem 1.48

Evaluate the following integrals:

(a) $\int (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2)\delta^3(\mathbf{r} - \mathbf{a})\, d\tau$, where $\mathbf{a}$ is a fixed vector, a is its magnitude, and the integral is over all space.

(b) $\int_{\mathcal{V}} |\mathbf{r} - \mathbf{b}|^2 \delta^3(5\mathbf{r})\, d\tau$, where $\mathcal{V}$ is a cube of side length 2, centered on the origin, and $\mathbf{b} = 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$.

(c) $\int_{\mathcal{V}} \left[r^4 + r^2(\mathbf{r} \cdot \mathbf{c}) + c^4\right] \delta^3(\mathbf{r} - \mathbf{c})\, d\tau$, where $\mathcal{V}$ is a sphere of radius 6 about the origin, $\mathbf{c} = 5\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$, and c is its magnitude.

(d) $\int_{\mathcal{V}} \mathbf{r} \cdot (\mathbf{d} - \mathbf{r})\delta^3(\mathbf{e} - \mathbf{r})\, d\tau$, where $\mathbf{d} = (1, 2, 3)$, $\mathbf{e} = (3, 2, 1)$, and $\mathcal{V}$ is a sphere of radius 1.5 centered at $(2, 2, 2)$.

Solution

Part (a)

$$
\begin{aligned}
\int (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2)\delta^3(\mathbf{r} - \mathbf{a})\, d\tau &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a})\delta^3(\mathbf{r} - \mathbf{a})\, d\mathbf{r} \\
&= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a} \\
&= a^2 + a^2 + a^2 \\
&= 3a^2
\end{aligned}
$$

Part (b)

$$
\begin{aligned}
\int_{\mathcal{V}} |\mathbf{r} - \mathbf{b}|^2 \delta^3(5\mathbf{r})\, d\tau &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (\mathbf{r} - \mathbf{b}) \cdot (\mathbf{r} - \mathbf{b})\delta^3(5\mathbf{r})\, d\mathbf{r} \\
&= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (\mathbf{r} - \mathbf{b}) \cdot (\mathbf{r} - \mathbf{b}) \left[\frac{1}{|5|^3}\delta^3(\mathbf{r})\right] d\mathbf{r} \\
&= \frac{1}{125}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (\mathbf{r} - \mathbf{b}) \cdot (\mathbf{r} - \mathbf{b})\delta^3(\mathbf{r})\, d\mathbf{r} \\
&= \frac{1}{125}(\mathbf{0} - \mathbf{b}) \cdot (\mathbf{0} - \mathbf{b}) \\
&= \frac{1}{125}(\mathbf{b} \cdot \mathbf{b}) \\
&= \frac{1}{125}(0^2 + 4^2 + 3^2) \\
&= \frac{1}{5}
\end{aligned}
$$

Part (c)

Observe that the magnitude of $\mathbf{c} = 5\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$ is $\sqrt{5^2+3^2+2^2} \approx 6.16$, which means the point $(5,3,2)$ lies outside the sphere that the volume integral is over. Therefore,

$$\begin{aligned}\int_{\mathcal{V}} \left[r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4\right]\delta^3(\mathbf{r}-\mathbf{c})\,d\tau &= \iiint\limits_{x^2+y^2+z^2\le 6^2} \left[r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4\right]\delta^3(\mathbf{r}-\mathbf{c})\,d\mathbf{r} \\ &= 0.\end{aligned}$$

Part (d)

Start by calculating the distance that the point $(3,2,1)$ is from $(2,2,2)$, the center of the sphere being integrated over.

$$\sqrt{(3-2)^2+(2-2)^2+(1-2)^2} = \sqrt{2} \approx 1.414$$

Since the radius of the sphere is 1.5, $(3,2,1)$ lies within it. Therefore,

$$\begin{aligned}\int_{\mathcal{V}} \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})\,d\tau &= \iiint\limits_{\substack{(x-2)^2+(y-2)^2\\+(z-2)^2\le 1.5^2}} \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})\,d\mathbf{r} \\ &= \iiint\limits_{\substack{(x-2)^2+(y-2)^2\\+(z-2)^2\le 1.5^2}} \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\left[\frac{1}{|-1|^3}\delta^3(\mathbf{r}-\mathbf{e})\right]d\mathbf{r} \\ &= \iiint\limits_{\substack{(x-2)^2+(y-2)^2\\+(z-2)^2\le 1.5^2}} \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{r}-\mathbf{e})\,d\mathbf{r} \\ &= \mathbf{e}\cdot(\mathbf{d}-\mathbf{e}) \\ &= \langle 3,2,1\rangle\cdot(\langle 1,2,3\rangle - \langle 3,2,1\rangle) \\ &= \langle 3,2,1\rangle\cdot\langle -2,0,2\rangle \\ &= 3(-2)+2(0)+1(2) \\ &= -4.\end{aligned}$$